

Taxicab Angles and Trigonometry

Kevin Thompson ^{*} & Tevian Dray [†]

Department of Mathematics, Oregon State University, Corvallis, OR 97331

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Abstract

A natural analogue to angles and trigonometry is developed in taxicab geometry. This structure is then analyzed to see which, if any, congruent triangle relations hold. A nice application involving the use of parallax to determine the exact (taxicab) distance to an object is also discussed.

^{*}Current address: North Arkansas College, 1515 Pioneer Drive, Harrison, AR 72601 / kthompson@northark.edu

[†]Send enquiries to: tevian@math.orst.edu

1 INTRODUCTION

Taxicab geometry, as its name might imply, is essentially the study of an ideal city with all roads running horizontal or vertical. The roads must be used to get from point A to point B; thus, the normal Euclidean distance function in the plane needs to be modified. The shortest distance from the origin to the point (1,1) is now 2 rather than $\sqrt{2}$. So, taxicab geometry is the study of the geometry consisting of Euclidean points, lines, and angles in \mathbb{R}^2 with the taxicab metric

$$d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|.$$

A nice discussion of taxicab geometry was given by Krause [1, 2], and some of its properties have been discussed elsewhere, including taxicab conic sections [3, 4, 5, 6], and the taxicab isometry group [7].

In this paper we will explore a slightly modified version of taxicab geometry. Instead of using Euclidean angles measured in radians, we will mirror the usual definition of the radian to obtain a taxicab radian (a t-radian). (A similar approach was used by Euler [8] to discuss the value of π for a class of generalized circles which includes the taxicab circle.) Using this definition, we will define taxicab trigonometric functions and explore the structure of the addition formulas from trigonometry. As applications of this new type of angle measurement, we will explore the existence of congruent triangle relations and illustrate how to determine the distance to a nearby object by performing a parallax measurement.

Henceforth, the label taxicab geometry will be used for this modified taxicab geometry; a subscript e will be attached to any Euclidean function or quantity.

2 TAXICAB ANGLES

There are at least two common ways of defining angle measurement: in terms of an inner product and in terms of the unit circle. For Euclidean space, these definitions agree. However, the taxicab metric is not an inner product since the natural norm derived from the metric does not satisfy the parallelogram law. Thus, we will define angle measurement on the unit taxicab circle which is shown in Figure 1.

Definition 2.1 A *t-radian* is an angle whose vertex is the center of a unit (taxicab) circle and intercepts an arc of (taxicab) length 1. The taxicab measure of a taxicab angle θ is the number of t-radians subtended by the angle on the unit taxicab circle about the vertex.

It follows immediately that a taxicab unit circle has 8 t-radians since the taxicab unit circle has a circumference of 8. For reference purposes the Euclidean angles $\pi/4$, $\pi/2$, and π in standard position now have measure 1, 2, and 4, respectively. The following theorem gives the formula for determining the taxicab measures of some other Euclidean angles.

Theorem 2.2 An acute Euclidean angle ϕ_e in standard position has a taxicab measure of

$$\theta = 2 - \frac{2}{1 + \tan_e \phi_e} = \frac{2 \sin_e \phi_e}{\sin_e \phi_e + \cos_e \phi_e}$$

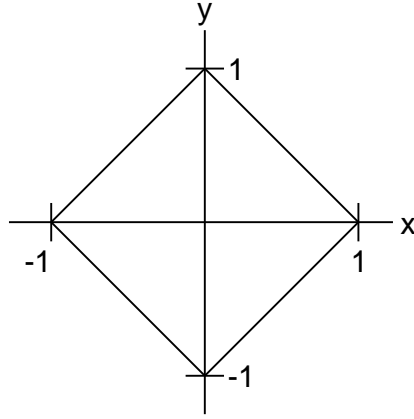


Figure 1: The taxicab unit circle.

Proof: The taxicab measure θ of the Euclidean angle ϕ_e is equal to the taxicab distance from $(1,0)$ to the intersection of the lines $y = -x + 1$ and $y = x \tan_e \phi_e$. The x-coordinate of this intersection is

$$x_0 = \frac{1}{1 + \tan_e \phi_e},$$

and thus the y-coordinate of P is $y_0 = -x_0 + 1$. Hence, the taxicab distance from $(1,0)$ to P is

$$\theta = 1 - x_0 + y_0 = 2 - \frac{2}{1 + \tan_e \phi_e}. \quad \square$$

Definition 2.3 *The reference angle of an angle ϕ is the smallest angle between ϕ and the x-axis.*

Theorem 2.2 can easily be extended to any acute angle lying entirely in a quadrant.

Corollary 2.4 *If an acute Euclidean angle ϕ_e with Euclidean reference angle ψ_e is contained entirely in a quadrant, then the angle has a taxicab measure of*

$$\begin{aligned} \theta &= \frac{2}{1 + \tan_e \psi_e} - \frac{2}{1 + \tan_e(\phi_e + \psi_e)} \\ &= \frac{2 \sin_e \phi_e}{(\cos_e(\phi_e + \psi_e) + \sin_e(\phi_e + \psi_e))(\cos_e \psi_e + \sin_e \psi_e)} \end{aligned}$$

This corollary implies the taxicab measure of a Euclidean angle in non-standard position is not necessarily equal to the taxicab measure of the same Euclidean angle in standard position. Thus, although angles are translation invariant, they are not rotation invariant. This is an important consideration when dealing with any triangles in taxicab geometry.

In Euclidean geometry, a device such as a cross staff or sextant can be used to measure the angular separation between two objects. The characteristics of a similar device to measure taxicab angles would be very strange to inhabitants of a Euclidean geometry; measuring the taxicab size of the same Euclidean angle in different directions would usually yield different results. Thus, to a Euclidean observer the taxicab angle measuring device must fundamentally change as it is pointed

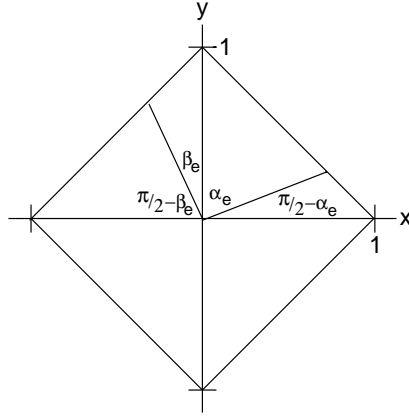


Figure 2: Taxicab right angles are precisely Euclidean right angles.

in different directions. Of course this is very odd to us since our own angle measuring devices do not appear to change as we point them in different directions.

The taxicab measure of other Euclidean angles can also be found. Except for a few cases, these formulas will be more complicated since angles lying in two or more quadrants encompass corners of the unit circle.

Lemma 2.5 *The taxicab measure of any Euclidean right angle is 2 t-radians.*

Proof: Without loss of generality, let θ be an angle encompassing the positive y-axis. As shown in Figure 2, split θ into two Euclidean angles α_e and β_e with reference angles $\pi/2 - \alpha_e$ and $\pi/2 - \beta_e$, respectively. Using Theorem 2.2, we see that

$$\begin{aligned} \theta &= \frac{2 \sin_e \alpha_e}{\cos_e \alpha_e + \sin_e \alpha_e} + \frac{2 \sin_e \beta_e}{\cos_e \beta_e + \sin_e \beta_e} \\ &= \frac{2 \sin_e \alpha_e}{\cos_e \alpha_e + \sin_e \alpha_e} + \frac{2 \cos_e \alpha_e}{\sin_e \alpha_e + \cos_e \alpha_e} \\ &= 2 \end{aligned}$$

since $\alpha_e + \beta_e = \pi/2$. \square

We now state the taxicab version of the familiar result for the length of an arc from Euclidean geometry and note that its proof is obvious since all distances along a taxicab circle are scaled equally as the radius is changed. This result will be used when we turn to congruent triangle relations and the concept of parallax.

Theorem 2.6 *The length s of the arc intercepted on a (taxicab) circle of radius r by the central angle with taxicab measure θ is given by $s = r\theta$.*

From the previous theorem we can easily deduce the following result.

Corollary 2.7 *Every taxicab circle has 8 t-radians.*

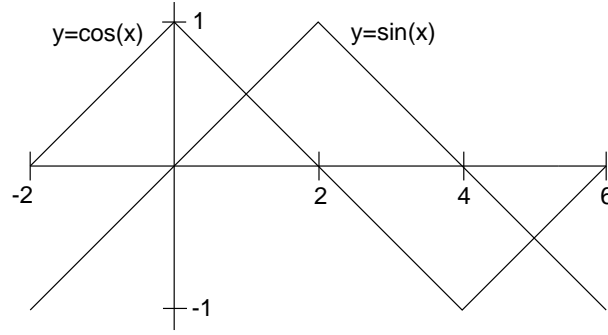


Figure 3: Graphs of the taxicab sine and cosine functions.

3 TAXICAB TRIGONOMETRY.

We now turn to the definition of the trigonometric functions sine and cosine in taxicab geometry. From these definitions familiar formulas for the tangent, secant, cosecant, and cotangent functions can be defined and results similar to those below can be obtained.

Definition 3.1 *The point of intersection of the terminal side of a taxicab angle θ in standard position with the taxicab unit circle is the point $(\cos_t \theta, \sin_t \theta)$.*

It is important to note that the taxicab sine and cosine values of a taxicab angle do not agree with the Euclidean sine and cosine values of the corresponding Euclidean angle. For example, the angle 1 t-radian has equal taxicab sine and cosine values of 0.5. The range of the cosine and sine functions remains $[-1, 1]$, but the period of these fundamental functions is now 8. It also follows immediately (from the distance function) that $|\sin_t \theta| + |\cos_t \theta| = 1$. In addition, the values of cosine and sine vary (piecewise) linearly with θ :

$$\cos_t \theta = \begin{cases} 1 - \frac{1}{2}\theta, & 0 \leq \theta < 2 \\ -3 + \frac{1}{2}\theta, & 2 \leq \theta < 4 \\ 1 - \frac{1}{2}\theta, & 4 \leq \theta < 6 \\ -3 + \frac{1}{2}\theta, & 6 \leq \theta < 8 \end{cases}, \quad \sin_t \theta = \begin{cases} \frac{1}{2}\theta, & 0 \leq \theta < 2 \\ 2 - \frac{1}{2}\theta, & 2 \leq \theta < 4 \\ -4 + \frac{1}{2}\theta, & 4 \leq \theta < 6 \\ 6 - \frac{1}{2}\theta, & 6 \leq \theta < 8 \end{cases}$$

Table 1 gives useful straightforward relations readily derived from the graphs of the sine and cosine functions which are shown in Figure 3. The structure of the graphs of these functions is similar to that of the Euclidean graphs of sine and cosine. Note that the smooth transition from increasing to decreasing at the extrema has been replaced with a corner. This is the same effect seen when comparing Euclidean circles with taxicab circles.

$\sin_t(-\theta) = -\sin_t \theta$	$\sin_t(\theta + 2) = \cos_t \theta$
$\cos_t(-\theta) = \cos_t \theta$	$\cos_t(\theta - 2) = \sin_t \theta$
$\sin_t(\theta - 4) = -\sin_t \theta$	$\sin_t(\theta + 8k) = \sin_t \theta, k \in \mathbb{Z}$
$\cos_t(\theta - 4) = -\cos_t \theta$	$\cos_t(\theta + 8k) = \cos_t \theta, k \in \mathbb{Z}$

Table 1: Basic Taxicab Trigonometric Relations

As discussed below, and just as in the standard taxicab geometry described in [2], SAS congruence for triangles does not hold in modified taxicab geometry. Thus, the routine proofs of sum and difference formulas are not so routine in this geometry. The first result we will prove is for the cosine of the sum of two angles. The formula given for the cosine of the sum of two angles only takes on two forms; the form used in a given situation depends on the locations of α and β . The notation $\alpha \in I$ will be used to indicate α is an angle in quadrant I and similarly for quadrants II, III, and IV.

Theorem 3.2 $\cos_t(\alpha + \beta) = \pm(-1 + |\cos_t \alpha \pm \cos_t \beta|)$ where the signs are chosen to be negative when α and β are on different sides of the x -axis and positive otherwise.

Proof: Without loss of generality, assume $\alpha, \beta \in [0, 8)$, for if an angle θ lies outside $[0, 8)$, $\exists k \in \mathbb{Z}$ such that $(\theta + 8k) \in [0, 8)$ and use of the identity $\cos_t(\theta + 8k) = \cos_t \theta$ will yield the desired result upon use of the following proof.

All of the subcases have a similar structure. We will prove the subcase $\alpha \in II, \beta \in III$. In this situation $6 \leq \alpha + \beta \leq 10$ and we take the negative signs on the right-hand side of the equation. Thus,

$$\begin{aligned}
1 - |\cos_t \alpha - \cos_t \beta| &= 1 - |1 - \frac{1}{2}\alpha - (-3 + \frac{1}{2}\beta)| \\
&= 1 - |4 - \frac{1}{2}(\alpha + \beta)| \\
&= \begin{cases} -3 + \frac{1}{2}(\alpha + \beta) , & 6 \leq \alpha + \beta < 8 \\ 5 - \frac{1}{2}(\alpha + \beta) , & 8 \leq \alpha + \beta \leq 10 \end{cases} \\
&= \cos_t(\alpha + \beta) \quad \square
\end{aligned}$$

Corollary 3.3 $\cos_t(2\alpha) = -1 + 2|\cos_t \alpha|$.

The curious case structure in Theorem 3.2 is due to the odd combinations of quadrants that determine which sign to choose. The reason for the sign change when α and β are on different sides of the x -axis lies in the fact that a corner of the cosine function is being crossed (i.e. different pieces of the cosine function are being used) to obtain the values of the cosine of α and β . Table 2 summarizes which form of $\cos_t(\alpha + \beta)$ should be used when.

	α	β
$\cos_t(\alpha + \beta) = -1 + \cos_t \alpha + \cos_t \beta $	same	quadrant
	I III	II IV
$\cos_t(\alpha + \beta) = 1 - \cos_t \alpha - \cos_t \beta $	I	III
	I	IV
	II	III
	II	IV

Table 2: Forms of $\cos_t(\alpha + \beta)$ and Regions of Validity

	α	β
$\sin_t(\alpha + \beta) = -1 + \sin_t \alpha + \cos_t \beta $	I	III
	I	IV
	II	II
	IV	IV
$\sin_t(\alpha + \beta) = 1 - \sin_t \alpha - \cos_t \beta $	I	I
	I	II
	II	III
	II	IV
	III	III
	III	IV

Table 3: Forms of $\sin_t(\alpha + \beta)$ and Regions of Validity

We can use Theorem 3.2 and the relations in Table 1 to establish a pair of corollaries.

Corollary 3.4 $\sin_t(\alpha + \beta) = \pm(-1 + |\sin_t \alpha \pm \cos_t \beta|)$ where the signs are chosen according to Table 3.

Proof: First, note $\sin_t \theta = \cos_t(\theta - 2)$. As with the cosine addition formula, all cases are proved similarly. We will assume $\alpha \in I$ and $\beta \in IV$. We have $\alpha - 2$ and β in the same quadrant, and thus

$$\begin{aligned}
\sin_t(\alpha + \beta) &= \cos_t((\alpha + \beta) - 2) \\
&= \cos_t((\alpha - 2) + \beta) \\
&= -1 + |\cos_t(\alpha - 2) + \cos_t \beta| \\
&= -1 + |\sin_t \alpha + \cos_t \beta| \quad \square
\end{aligned}$$

Corollary 3.5 $\sin_t(2\alpha) = -1 + 2|\cos_t(\alpha - 1)|$

Proof: $\sin_t(2\alpha) = \cos_t(2\alpha - 2) = \cos_t(2(\alpha - 1)) = -1 + 2|\cos_t(\alpha - 1)| \quad \square$

4 CONGRUENT TRIANGLES

In Euclidean geometry we have many familiar conditions that ensure two triangles are congruent. Among them are SAS, ASA, and AAS. In modified taxicab geometry the only condition that ensures two triangles are congruent is SASAS. One example eliminates almost all of the other conditions.

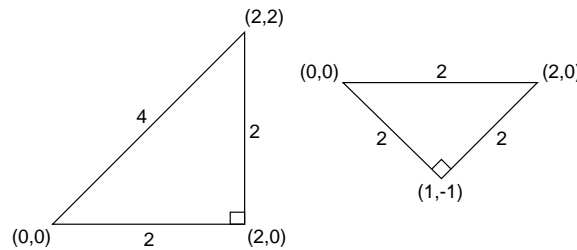


Figure 4: Triangles satisfying ASASA that are not congruent.

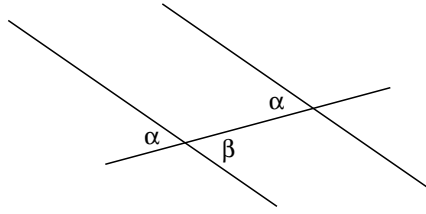


Figure 5: Alternate interior angles formed by parallel lines and a transversal are congruent.

Consider the two triangles shown in Figure 4. The triangle formed by the points $(0,0)$, $(2,0)$, and $(2,2)$ has sides of lengths 2, 2, and 4 and angles of measure 1, 1, and 2 t-radians. The triangle formed by the points $(0,0)$, $(2,0)$, and $(1,-1)$ has sides of length 2 and angles of measure 1, 1, and 2. These two triangles satisfy the ASASA condition but are not congruent. This also eliminates the ASA, SAS, and AAS conditions as well the possibility for a SSA or AAA condition.

The triangle formed by the points $(0,0)$, $(0.5,1.5)$, and $(1.5, 0.5)$ has sides of length 2 and angles 1, 1.5, and 1.5 t-radians. Thus, it satisfies the SSS condition with the second triangle in the previous example. However, the angles of these triangles are not congruent. Hence, the SSS and SSSA conditions fail.

The last remaining condition, SASAS, actually does hold. Its proof relies on the fact that even in this geometry the sum of the angles of a triangle is a constant 4 t-radians, which in turn relies on the fact that, given parallel lines and a transversal, alternate interior angles are congruent. We begin by noting that opposite angles are congruent. This leads immediately to the following result.

Lemma 4.1 *Given two parallel lines and a transversal, the alternate interior angles are congruent.*

Proof: Using Figure 5 translate α along the transversal to become an angle opposite β . By the note above, α and β are congruent. \square

Theorem 4.2 *The sum of the angles of a triangle in modified taxicab geometry is 4 t-radians.*

Proof: Given the triangle in Figure 6, we can translate the angle γ from Q to R and by the congruence of alternate interior angles conclude the sum of the angles of the triangle is 4 t-radians. \square

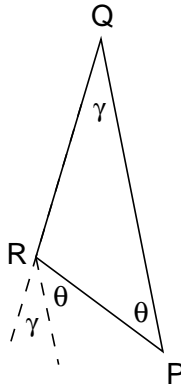


Figure 6: The sum of the angles of a taxicab triangle is always 4 t-radians.

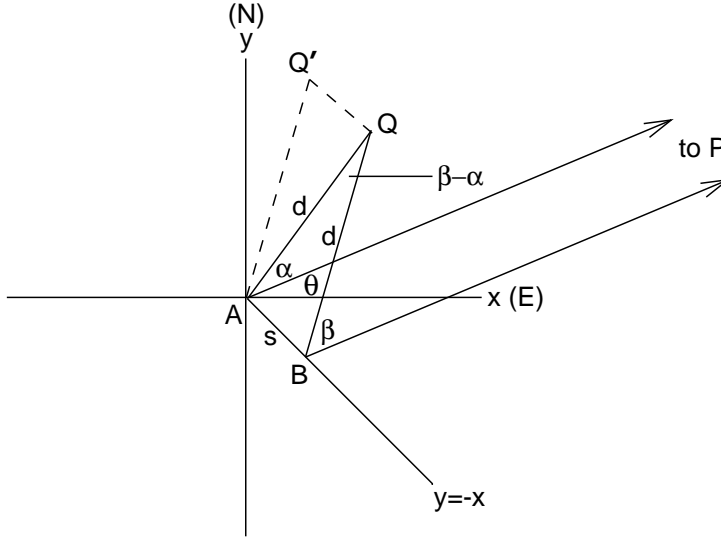


Figure 7: A parallax diagram in taxicab geometry.

Therefore, given two triangles having all three sides and any two angles congruent, the triangles must be congruent. However, as we have seen, this is the only congruent triangle relation in taxicab geometry.

5 PARALLAX

Parallax, the apparent shift of an object due to the motion of the observer, is a commonly used method for estimating the distance to a nearby object. The method of stellar parallax was used extensively to find the distances to nearby stars in the 19th and early 20th centuries. We now wish to explore the method and results of parallax in taxicab geometry and examine how these differ from the Euclidean method and results. We will discover that the taxicab method yields the same formula commonly used in the Euclidean case with the exception that the taxicab formula is exact.

Suppose that as a citizen of Modified Taxicab-land you wish to find the distance to a nearby object Q in the first quadrant, and that there is also a distant reference object P “at infinity” essentially in the same direction as Q with reference angle θ (Figure 7). The distant reference object should be far enough away so that it appears stationary when you move small distances. We may assume without loss of generality that the object Q does not lie on either axis, for if it did, we could move a small distance to get the object in the interior of the first quadrant.

Initially standing at A , measure the angle α between Q and P using the taxicab equivalent of a cross staff or a sextant. Now, for reasons to be apparent later, you should move a small distance (relative to the distance to the object) in such a way that the distance to the object does not change. This can be accomplished by moving in either of two directions, and, provided you move only a small distance, the object remains in the interior of the first quadrant. Furthermore, exactly one of these directions results in the angle between Q and P being increased, so that the situation depicted in Figure 7 is generic.

You have therefore moved from A to B in one of the following directions: NW, NE, SW, or SE. Now measure the new angle β between the two objects. With this information we can now find

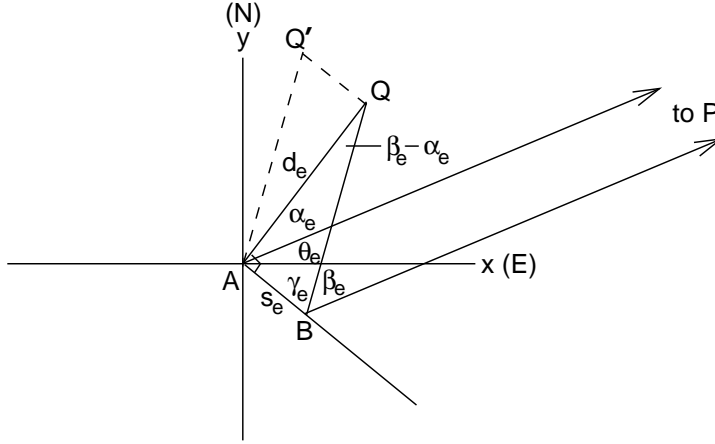


Figure 8: A parallax diagram in Euclidean geometry.

the taxicab distance to the object Q . Construct the point Q' such that $\overline{QQ'}$ is parallel to \overline{AB} and $\ell(\overline{QQ'}) = \ell(\overline{AB}) = s$. The angle $\angle PAQ'$ has measure β since it is merely a translation of $\angle QBP$. Thus, $\angle QAQ'$ has measure $\beta - \alpha$. Now, the lengths of \overline{AQ} and $\overline{AQ'}$ are equal since the direction of movement from A to B was shrewdly chosen so that the distance to the object remained constant. Since translations do not affect lengths, this implies AQ and AQ' have equal lengths. Hence, the points Q and Q' lie on a taxicab circle of radius d centered at A . Using the formula for the length of a taxicab arc in Theorem 2.6,

$$d = \frac{s}{\beta - \alpha} \quad (1)$$

where s and d are taxicab distances and $\beta - \alpha$ is a taxicab angle. This formula is identical to the Euclidean distance *estimation* formula with d and s Euclidean distances and $\beta - \alpha$ an Euclidean angle. However, as we shall now see, the commonly used Euclidean version is truly an approximation and not an exact result. This realization is necessary to logically link the commonly used Euclidean formula and the taxicab formula.

Using Figure 7 but with all distances and angles now Euclidean, we isolate $\triangle QAB$. Using the law of sines and the fact that $\gamma = 3\pi/4 - (\beta_e + \theta_e)$, we have

$$d_e = \frac{s_e(\cos_e(\beta_e + \theta_e) + \sin_e(\beta_e + \theta_e))}{\sqrt{2}\sin_e(\beta_e - \alpha_e)} \quad (2)$$

This formula can be simplified by moving from A to B in a direction perpendicular to the line of sight to Q from A rather than in one of the four prescribed directions above. In this case $m(\angle QBA) = \pi/2 - (\beta_e - \alpha_e)$ (Figure 8). Thus, the law of sines gives the Euclidean parallax formula

$$d_e = \frac{s_e}{\tan_e(\beta_e - \alpha_e)}$$

If we now apply the approximation $\tan_e(\beta_e - \alpha_e) \approx (\beta_e - \alpha_e)$ for small angles we obtain the commonly used Euclidean parallax formula

$$d_e \approx \frac{s_e}{\beta_e - \alpha_e}.$$

which is not exact.

It is interesting to note the quite different movement requirements in the two geometries needed to obtain the best possible approximations of the distance to the object. This difference lies in the methods of keeping the distance to the object as constant as possible. In the Euclidean case, moving small distances on the line tangent to the circle of radius d centered at the object (i.e. perpendicular to the radius of this circle) essentially leaves the distance to the object unchanged. In taxicab geometry, moving in one direction along either $y = x$ or $y = -x$ keeps the distance to the object exactly unchanged.

We are now in a position to justify the link between results (1) and (2). Since the line segment $\overline{QQ'}$ of taxicab length s lies on a taxicab circle, $s = \sqrt{2}s_e$. The distance d to the object is given by $d = d_e(\cos_e(\alpha_e + \theta_e) + \sin_e(\alpha_e + \theta_e))$ since the Euclidean angle between the line of sight \overline{AQ} and the x-axis is $(\alpha_e + \theta_e)$. Using Corollary 2.4 with $\phi = (\beta_e - \alpha_e)$ and $\psi = (\alpha_e + \theta_e)$, the taxicab measure of $\beta - \alpha$ is given by

$$\beta - \alpha = \frac{2 \sin_e(\beta_e - \alpha_e)}{(\cos_e(\beta_e + \theta_e) + \sin_e(\beta_e + \theta_e))(\cos_e(\alpha_e + \theta_e) + \sin_e(\alpha_e + \theta_e))}.$$

Using these substitutions, formula (1) becomes formula (2).

6 CONCLUSION

With this natural definition of angles in taxicab geometry, some of the same difficulties arise as with Euclidean angles in taxicab geometry. Congruent triangles are few and far between. Only with the strictest requirements, namely that all three sides and two angles are congruent, are we able to conclude that two triangles must be congruent.

In addition to creating a natural definition of angles and trigonometric functions, we have also unwittingly created an environment in which a parallax method can be used to determine the exact distance to a nearby object rather than just an approximation. This is not too surprising a result since there exist directions in which one can travel without the distance to an object changing. With this result, and taxicab and Euclidean angle measuring instruments, the exact Euclidean distance to the object can now be found (up to measurement error of course). The trick is to build your own taxicab cross staff or sextant.

ACKNOWLEDGMENTS

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